Day 20
Wednesday, May 13, 2020 2:00 PM

Review.

Thm. Let H be a bipartite on (A,B) such that $\forall a \in A$ has degree at most r. Then $ex(n,H) \leq C n^{2-1/r}$.

Rmk. This bound can be tight for $H = K_{r,s}$ where $s > (r-1)!$

$\boxed{\text{Conjecture}}$. Let H be a $K_{r,r}$-free bipartite graph on (A,B) such that $\forall a \in A$ has degree at most r. Then $\exists\, c, C > 0$ s.t. $\boxed{ex(n,H) \leq C n^{2-1/r-c}}$ $\Bigg\}$ $\boxed{\text{open for } r \geq 3}$

(*) Today, we will show this conjecture holds for $r=2$.

$\boxed{\text{Thm 1}}$ Let H be a $K_{2,2}$-free bipartite graph on bipartition (A,B) such that each vertex in A has degree at most 2. Then there exist $c, C > 0$ depending on H such that

$$ex(n,H) \leq C n^{3/2 - c}$$

Def. For a graph H, the k-subdivision $H^{(k)}$ of H is a graph obtained from H by replacing each edge ab of H with an internally disjoint path P_{ab} of length $k+1$ with endpoints a and b, where all such paths P_{ab} are mutually internally disjoint.

e.g. $H =$  $H^{(1)} =$ 

K_4

$H^{(2)} =$ 

We point out that all $H^{(k)}$ are bipartite for all graphs H. (for odd $k \geq 1$)

Thm 2 (Conlon-Lee, Janzer '19) For all $t \geq 3$, there exists $c_t > 0$ such that

$$ex(n, K_t^{(1)}) = O(n^{3/2 - c_t})$$

$\boxed{c_t = \frac{1}{4t-6}}$

We observe that Thm 2 can imply Thm 1. (?)

[Because any H in Thm 1 is contained in $K_{|A|}^{(1)}$.
$\Rightarrow ex(n,H) \leq ex(n, K_{|A|}^{(1)})$]

A graph G is called K-regular if $\Delta(G) \leq K \cdot \delta(G)$.

Lemma 1 (Erdős-Simonovits; Jiang, Bukh-Jiang, Conlon-Lee)
For all $0 < \alpha < 1$, there exist constants $\beta, K > 0$ such that for all $C > 0$ and sufficiently large n, every n-vtx graph G with at least $C\boxed{n^{1+\alpha}}$ edges has a subgraph G' satisfying

(1) G' is $\boxed{K\text{-regular}}$ and bipartite with two parts of sizes differing by a factor at most 2,

(2) $v(G') \geq n^{\beta}$, and

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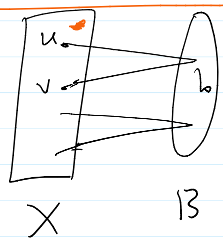
(2) $v(G') \geq n^{\beta}$, and

(3) $\dfrac{e(G')}{v(G')^{1+\alpha}} \geq \dfrac{1}{10} \cdot \dfrac{e(G)}{v(G)^{1+\alpha}}$ $\quad \boxed{\Longrightarrow e(G') \geq \frac{c}{10} \cdot n^{1+\alpha}}$

We will not give a detailed proof for this Lemma.

Def. Fix t. For $u, v \in V(G)$, we say the pair $\{u,v\}$ is light, if $1 \leq |N(u) \cap N(v)| < \binom{t}{2}$, and is heavy if $|N(u) \cap N(v)| \geq \binom{t}{2}$.

Lemma 2. Let G be a $K_t^{(1)}$-free bipartite graph with bipartition $X \cup B$, where $d(x) \geq \delta$ for all $x \in X$ and $\boxed{|X| \geq \frac{4|B|t}{\delta}}$. Then there exists $u \in X$ in $\Omega(\delta^2 |X| / |B|)$ light pairs in X.



$e(G) \geq \delta |X|$

Pf. Let $S = \left\{ (\{u,v\}, b) : \begin{array}{l} b \in B, \\ u,v \in X \cap N(b) \end{array} \right\}$

We see that

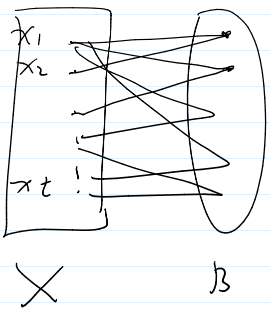
$$|S| = \sum_{b \in B} \binom{d(b)}{2} \geq |B| \binom{e(G)/|B|}{2}$$

$$\geq \frac{|B|}{4} \left(\frac{\delta |X|}{|B|} \right)^2 = \frac{\delta^2 |X|^2}{4|B|}.$$

Let $B^* = \{ b \in B : d(b) \geq 2t \}$.

Since $\displaystyle\sum_{b \in B \setminus B^*} \binom{d(b)}{2} \leq 2t^2 \cdot |B| \leq \frac{\delta^2 |X|^2}{8|B|}$

$$\Longrightarrow \boxed{\sum_{b \in B^*} \binom{d(b)}{2} \geq \frac{\delta^2 |X|^2}{8|B|} \qquad (1)}$$



claim: No t vertices in X, any pair of which is heavy

Pf. Suppose say $x_1, x_2, \ldots, x_t$ s.t. $\{x_i, x_j\}$ is heavy

i.e. $|N(x_i) \cap N(x_j)| \geq \binom{t}{2}$.

It is easy to see that there exists a $K_t^{(1)}$ in G, a contradiction $\blacksquare$

- Consider $b \in B^*$. Any pair in $N(b)$ is either light or heavy

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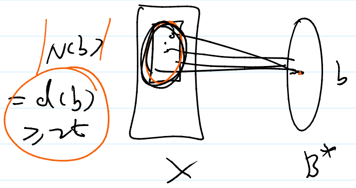

By Turán's Thm and the claim, the number of heavy pairs in $N(b)$ is at most $e(T_{t-1}(d(b)))$.

$\Rightarrow$ $\forall b \in B^*$, there are at least

$$\binom{d(b)}{2} - e(T_{t-1}(d(b))) \geq \binom{d(b)}{2} - \binom{t-1}{2}\left(\frac{d(b)}{t-1}\right)^2$$

$$\geq \frac{1}{2(t-1)} d(b)^2 - \frac{1}{2} d(b) \geq \Omega(d(b)^2)$$

light pairs in $N(b)$

If we sum over all $b \in B^*$, then

$$\# \triangleq \#(\{u,v\}, b) \geq \sum_{b \in B^*} \Omega(d(b)^2)$$

where $\{u,v\}$ is light in $N(b)$ & $b \in B^*$

$$\overset{①}{\geq} \Omega\left(\frac{\delta^2 |X|^2}{|B|}\right).$$

Since $\{u,v\}$ is light, we get

$$\# \text{ light pairs in } X \geq \frac{\#}{\binom{t}{2}} \geq \Omega\left(\frac{\delta^2 |X|^2}{|B|}\right)$$

$\Rightarrow$ $\exists$ a vertex $u \in X$ which is in at least

$\Omega\left(\frac{\delta^2 |X|}{|B|}\right)$ light pairs in X. ☒

Pf of Thm 2. We give a proof due to Janzer, who proved that $c_t = \frac{1}{4t-6}$ for $t \geq 3$.

$\boxed{t=3}$: $K_3^{(1)} = C_6$,

$ex(n, C_6) = \Theta(n^{4/3})$ is tight; perhaps $c_t = \frac{1}{4t-6}$ is tight for all $t \geq 3$?

Let G be a $K_t^{(1)}$-free graph on n vertices & with at least $D \cdot n^{3/2 - c_t} = D\, n^{1+\alpha}$ $\left(\alpha = \frac{t-2}{2t-3}\right)$ edges.

By Lemma 1 (with α), there exists a $G' \subseteq G$

which is K-regular and bipartite w/ parts $A \cup B$.

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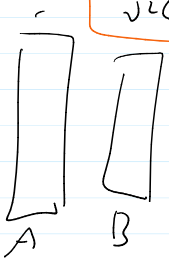
which is $\boxed{K\text{-regular}}$ and bipartite on parts $A \cup B$,
such that $\boxed{\frac{e(G')}{v(G')^{1+\alpha}} \geq \frac{1}{10} \frac{e(G)}{v(G)^{1+\alpha}}}$ & $v(G')$ is large,

$G' =$ 

$\frac{1}{2}|B| \leq |A| \leq 2|B|$

⊛ If $\delta(G') \leq C \cdot (v(G'))^{\alpha}$
then $\Delta(G') \leq K \cdot C \cdot (v(G'))^{\alpha}$
$\Rightarrow e(G') \leq KC \cdot (v(G'))^{1+\alpha}$
$\Rightarrow e(G) \leq 10KC \cdot n^{1+\alpha}$ $\boxed{\text{done}}$.

Therefore, $\delta \triangleq \delta(G') > C\left(v(G')\right)^{\alpha} = C \cdot \left(v(G')\right)^{(t-2)/(2t-3)}$

Our plan is to find t vertices $u_1, u_2, \dots, u_t$ in A
such that • $\{u_i, u_j\}$ is light $\forall\, 1 \leq i < j \leq t$, ✓
• u_i, u_j, u_k have NO common neighbors for all distinct i, j, k ✓

If so, then we can find a $K_t^{(l)}$ in G easily.

We will do so by repeatedly using lemma 2 on a stronger hypothesis: for each $1 \leq i \leq t$, there exists $A = X_1 \supseteq X_2 \supseteq X_3 \cdots \supseteq X_i$ and $u_1 \in X_1, u_2 \in X_2, \dots, u_i \in X_i$ s.t.

(a) u_j is in at least $\Omega\left(\delta^2 |X_j| / v(G')\right)$ light pairs in X_j for $\forall\, 1 \leq j \leq i-1$.

(b) u_j is light to every vertex w in X_{j+1} $\forall\, 1 \leq j \leq i-1$ $\Leftrightarrow$ $\{u_j, w\}$ is light

(c) NO 3 of $v_1, \dots, v_i$ have common neighbors

(d) $|X_{j+1}| = \Omega\left(\delta^2 |X_j| / v(G')\right)$ $\forall\, 1 \leq j \leq i-1$

This holds clearly for $i=1$ by choosing u_1 to be the vertex found by lemma 2 when applied to $A \cup B$.

Now suppose we have constructed this for $i-1$:

$A = X_1 \supseteq X_2 \supseteq \cdots \supseteq X_{i-1}$ with $u_j \in X_j$ $\forall\, j \leq i-1$

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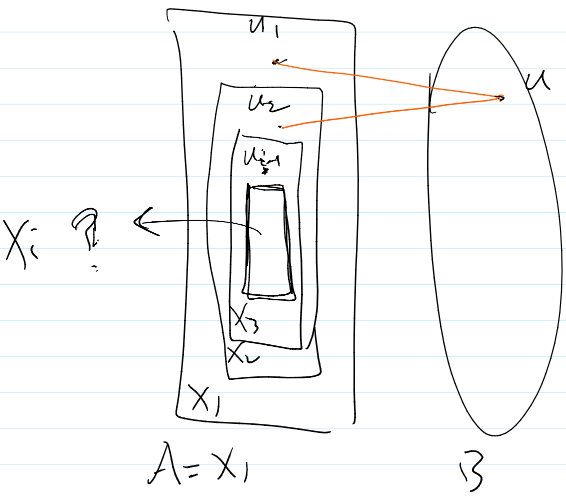
$A = X_1 \supseteq X_2 \supseteq \cdots \supseteq X_{i-1}$ with $u_j \in X_j$, $\forall j \le i-1$.



Let $Y_i = \{ y \in X_{i-1} : \{y, u_{i-1}\} \text{ is light} \}$

By (c), $|Y_i| \geq \delta^2 |X_{i-1}| / v(G')$

Consider any $u_j u_\ell$, $j, \ell \le i-1$, take any common neighbor u of them and delete $N(u)$ from Y_i

There are $\binom{i-1}{2}$ pairs $u_j u_\ell$ & there are at most $\binom{t}{2}$ many choices of u for each $u_j u_\ell$ & there are at most $k \delta$ vtx in $N(u)$.

$\implies$ # vtx deleted in this around is at most

$$\le \binom{i-1}{2} \binom{t}{2} k\delta = O(\delta).$$

As long as $\boxed{|Y_i| \geq \delta^2 |X_{i-1}| / v(G') \geq \Omega(\delta)}$,

then we can get a $X_i \subseteq Y_i$ of size at least

$\boxed{|X_i| \geq \Omega\left(\frac{\delta^2}{v(G')}\right) |X_{i-1}|}$, which satisfies (c).

This is true, because $i \le t$

$$\left(\frac{\delta^2}{v(G')}\right)^{i-1} |A| \gtrsim \left(\frac{\delta^2}{n}\right)^{i-1} n \geq \Omega(\delta)$$

$$\iff \delta^{2t-3} \geq n^{t-2}$$

This shows that the algorithm can keep going until we have $X_1 \supseteq X_2 \supseteq \cdots \supseteq X_t$

... $u_i \in X_i$, $1 \le i \le t$.

going until we have $X_1 - X_t$ ...

and vertices $u_j \in X_j$, $1 \le j \le t$.

This is clear from this to see

- $\{u_i, u_j\}$ is light, $\forall i \neq j$
- any 3 of $u_1, u_2, \dots, u_t$ have NO common neighbors